Heawood Conjecture and the map coloring theorem

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1 Introduction

One of the most well known results in graph theory is the four color theorem, which states that any planar graph's chromatic color is four or less. This theorem has a long and interesting history, however it is only a single case in the much larger map color theorem or Heawood conjecture. The map color theorem gives an upper bound on the chromatic coloring for surfaces with a given genus. Interestingly, the map color theorem was proven for all cases other than the planer case first.

In this post, we will explore an overview of the history of the map color theorem, both how it follows the four color theorem and how it branches off on its own. Then we will build some preliminary information needed to prove part of the map color theorem. Once the preliminaries are set I want to give a proof overview of the Heawood inequality which is the upper bound of the map color theorem. Finally, some concluding remarks on the map color theorem will be given.

2 History

2.1 Percy John Heawood

Percy John Heawood (September 1861 – January 1955) was British mathematician who focused on graph coloring. Some of his notable achievements were in 1890 finding a flaw with Alfred Kempe proof of the four color theorem which had been considered valid for over a decade, and, also in 1890, what became known as the Heawood conjecture which was a formula for the upper bound on coloring higher genus (informally the number of holes a surface has) surfaces.

In the original form of the conjecture Heawood left out the planar case. It was an open question at the time whether four or five colors was nessesary and his formula predicted four to be the upper bound.

Heawood proved the upper bound, but it would take another 80 years for this to be proven to be the upper and lower bound for maximal chromatic coloring of these surfaces.

2.2 Twelve cases thirteen years

Because Heawood was able to prove that his formula was an upper bound the question to show equality became finding complicated constructions as counter examples for any lower values. Due to symmetries of the surfaces it was quickly shown that the problem could be broken into 12 cases. Gerhard Ringel, John Youngs, C.M. Terry, W. Gustin, and Lloyd Welch would construct these counter examples in 13 years over the course of seven papers.

Some exceptions arised and special cases were needed for genus 18, 20, 23, 30, 35, 47, and 59. These were quickly handled in only two years and in 1968 the map color theorem was considered proven (noting that this still excluded the planar case which would take another 8 years to prove). From their major contributions into the cases the map color theorem is also known as the Ringel–Youngs theorem.

2.3 Four color theorem

The four color theorem is a older problem that ended up taking a lot longer to prove then the Heawood conjecture. What was interesting is in the Heawood conjecture proving the upper bound was quite easy and the difficulty was showing that this upper bound was sufficient. However for the four color theorem the opposite was known. A lower bound of four colors was known almost immediately, but showing this to be sufficient was extremely hard.

Once the four color theorem was proven the map color theorem could be extended to include the planar case since the formula correctly predicted that four colors would be needed.

2.4 Klein Bottle

When Heawood made his original conjecture his focus was on, orientable two-dimensional manifolds two sided surfaces. However, his conjecture was easily shown to work for general surfaces using the Euler characteristic instead of the genus of a surface.

In 1933 Philip Franklin showed that this generalization does not work for any surface. He showed that the chromatic coloring of a graph on a Klein bottle is always six or less. This raised doubts about the validity of the Heawood conjecture and specifically the four color theorem, which had an open question of whether four or five colors were needed.

This turned out to be the only exception to the map color theorem.

3 Preliminaries to prove

3.1 Graph theory

I am going to assume a basic understanding of graph theory and not cover most of the definitions, but some important ones are as follows.

Definition 3.1. *The* ***chromatic color of a graph*** *in this paper refers to the minimum number of colors needed to color the vertices of a graph such that no two vertices who share an edge have the same color.*

Definition 3.2. *For a graph $G = (V, E)$ the quantity $V - E + F$ is called the Euler characteristic of a graph.*

Definition 3.3. *Critical subgraph is a minimum subgraph that has the same chromatic color as the original graph.*

3.2 Topology

Definition 3.4. *A* ***topology*** *on a set X is a collection $\mathscr{T}$ of subsets of X having the following properties:*

1. *$\emptyset$ and X are in $\mathscr{T}$.*
2. *The union of the elements of any subcollection of $\mathscr{T}$ is in $\mathscr{T}$.*
3. *The intersection of the elements of any finite subcollection of $\mathscr{T}$ is in $\mathscr{T}$.*

A set X for which a topology $\mathscr{T}$ has been specified is called a ***topological space****.*

Definition 3.5. *A* ***homeomorphism*** *is a bijective and continuous function between topological spaces that has a continuous inverse function.*

Definition 3.6. *If f and f' are continuous maps of the space X into the space Y, we say that f is homotopic to f' if there is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ and for each x. (Here $I = [0, 1]$.) The map F is called a homotopy between f and f' . If f is homotopic to f', we write $f \cong f'$. If $f \cong f'$ and f' is a constant map, we say that f is nulhomotopic.*

Definition 3.7. *A surface S in the Euclidean space $\mathbb{R}^3$ is* ***orientable*** *if a two-dimensional figure cannot be moved around the surface and back to where it started so that it looks like its own mirror image.*

Most surfaces that can be drawn in the dimension that you would expect them to live are orientable. Examples include spheres, tori, and discs.

Non-orientable surfaces often have interesting ways of combining together which leads to strange properties. A well known example is the Klein bottle which is a four dimensional surface that has only one side and is of genus one.

Definition 3.8. *The genus of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected.*

The common way to interpret this is the number of holes a surface has. The more holes the higher the genus since we can cut the edge of each hole leaving a connected surface.

Combining genus with homotopies we can form classes of surfaces since if two surfaces have the same genus, then there exists a homotopy between them.

There is another definition of genus that deals with the Euler characteristic of a graph which allows for the genus of non-orientable surfaces which is

Definition 3.9. *Alternatively, the genus of a closed surface is*

$$\chi = 2 - 2g,$$

where χ is the Euler characteristic and g is the genus.

Definition 3.10. *The **klein bottle** is a closed non-orientable surface of Euler characteristic 0 that has no inside or outside.*

Interesting to note about the klein bottle, is that while it needs four dimensions to be constructed it is only a two dimensional manifold. Or at any point locally it looks like two dimensional Euclidean space.

Definition 3.11. *An **embedding of a graph** G on a surface S is a representation of G on S in which points of S are associated with vertices and simple arcs (homeomorphic images of $[0, 1]$) are associated with edges in such a way that:*

- *The endpoints of the arc associated with an edge e are the points associated with the end vertices of e.*
- *No arcs include points associated with other vertices.*
- *Two arcs never intersect at a point which is interior to either of the arcs.*

Definition 3.12. *The **chromatic color of a surface** S is the maximum number of colors needed to color any graph that can be embedded onto S.*

Give an example of a 7 coloring of a torus

4 Proof of the Heawood inequality

Theorem 4.1. *If S is a closed surface with Euler characteristic $E(S) \neq 2$ (different from the sphere), then the chromatic color of S is*

$$\chi(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24E(S)}}{2} \right\rfloor .$$

The proof of this theorem relies on several ideas out of scope of this paper. However the process is first noting that the the chromatic color of a surface is the maximum of the chromatic coloring of a graph that can be embedded onto it. We can then assume that the graph G is critical (if not take the subgraph to make it critical).

Next taking that graph we can do algebra with its Euler characteristic and inequalities for the number of edges and vertices to get

$$\chi^2 - \chi \leq 6\chi - 6E,$$

where χ is the chromatic coloring of G and E is the Euler characteristic.

Finally we can use the quadratic formula to get this in the form

$$\left(\chi - \frac{7+\sqrt{49-24E}}{2}\right)\left(\chi - \frac{7-\sqrt{49-24E}}{2}\right) \leq 0.$$

Which noting that $\chi \geq 1$ gives that the second product with $7-\sqrt{49-24E}$ does not make sense. So we are left with

$$\chi \leq \frac{7+\sqrt{49-24E}}{2}.$$

The rest of the proof is dealing with edge cases and clarifying jumps. I think this proof is interesting because once the inequalities are set it becomes a matter of algebra and dealing with cases based on what graphs can exist.

5 Concluding remarks

Overall, I found this to be a really interesting topic. I was especially interesting in how proving the cases other then the plane were far easier then the plane itself. This mainly comes from the Euler characteristic of the plane being 2 and having difficulties using inequalities in that case because you run into divide by 0 problems.

Some topics that I wanted to dive further into but I ran out of time were why is the Klein bottle the only exception to the rule? It seems like once an exception like this is found you could construct a process for making more surfaces like the Klein bottle in higher dimensions. The other topic would be why 12 cases for the full proof. Somehow it was quickly found that we could reduce all higher dimensional genus graphs into 12 residual graphs. The question of why 12 is something I want to know more about.

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