The cone of polynomials that preserve nonnegative matrices

Benjamin J. Clark

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Nonnegative Inverse Eigenvalue Problem (NIEP)

Given a finite list $\Lambda = \{s_1, \ldots, s_n\}$ of complex numbers, the NIEP asks for necessary and sufficient conditions such that Λ is the spectrum of an n-by-n entrywise-nonnegative matrix.

Background on the problem

- In pursuit of a solution to the NIEP, Loewy and London in 1978 posed the problem of characterizing all polynomials that preserve all nonnegative matrices of a fixed order.
- Clark and Paparella in 2021 showed the set of polynomials that preserve nonnegative matrices form a convex non-polyhedral cone with respect to the coefficients of the polynomials.
- Lowey in 2023 considered restricting the degree of the polynomials and showed that polynomials of degree 4 form a non-polyhedral cone with respect to 2 by 2 nonnegative matrices.

Notation

- M_n denotes the set of all n-by-n real matrices.
- $A \in \mathrm{M}_n$ is nonnegative, denoted $A \geq 0$, if it is entry-wise nonnegative. Similar definition for a positive matrix.
- $\mathrm{M}_n^{\geq 0}$ and M_n^{+} denotes the sets of all n-by-n real nonnegative matrices and real positive matrices.
- $\mathbb{R}[t]$ is defined as all polynomials of a finite degree with real coefficients
- The first n terms of a polynomial are the terms indexed by $\{0, 1, ..., n-1\}$. Similar definition for the last n terms.
- The set $\langle m \rangle$ are the natural numbers from 1 to m inclusive. The set $\langle m \rangle_0$ also includes 0.

Sets of polynomials that preserve nonnegative matrices

The set of polynomials that preserve nonnegative matrices of a given order is defined as

$$\mathcal{P}_n := \{p \in \mathbb{R}[t] \mid p(A) \geq 0, \forall A \in \mathrm{M}_n^{\geq 0}\}.$$

Also define the set

$$\mathcal{P}_{n,m} := \{p \in \mathcal{P}_n \mid \text{degree}(p) \leq m\},$$

where we restrict the degree of the polynomials.

Nonnegative polynomials

Lemma

If $p \in \mathbb{R}[x]$ such that all the coefficients of p are nonnegative, then $p \in \mathcal{P}_n$ for every $n \geq 1$.

Question

When can the coefficients of the polynomials be negative?

Remainder polynomials

Let $n \in \mathbb{N}$ and $r \in \langle n-1 \rangle_0$. If

$$\mathcal{I}_{(r,n)} := \{k \in \mathbb{N} \mid k \equiv r \bmod n\},$$

then the polynomial

$$p_{(r,n)}(x) := \sum_{k \in \mathcal{I}_{(r,n)}} a_k x^k,$$

is called the r mod n *part of* p or the r mod n remainder polynomial.

Circulant matrices

- A circulant is a matrix of the form

$$A = \operatorname{circ}(a_0, a_1, \ldots, a_{n-1}) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}.$$

- There is a special type of circulant called the fundamental circulant or push circulant which has the following form

$$C := \operatorname{circ}(0, 1, 0, ..., 0) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

Properties of the fundamental circulant

- Any circulant matrix can be decomposed into a polynomial made with the fundamental circulant. Let $A = \text{circ}(a_0, a_1, \ldots, a_{n-1})$, then

$$A = p_A(C) = a_{n-1}C^{n-1} + a_{n-2}C^{n-2} + \cdots + a_1 C + a_0 I.$$

- The fundamental circulant forms a cycle of length n, that is $C^{nq+r} = C^r$ for any $r \in \langle n-1 \rangle$.

Polynomials and the fundamental circulant

Using the fact that the fundamental circulant forms cycles, we can "decompose" our polynomial into it's n remainder polynomials.

$$\begin{aligned} p(xC) &= \sum_{j=0}^{m} a_j x^j C^j \\ &= \sum_{j=0}^{m} a_j x^j C^{j \bmod n} \\ &= \operatorname{circ}\left(\sum_{j \in \mathcal{I}_{(0,n)}}^{n-1} a_j x^j, \sum_{j \in \mathcal{I}_{(1,n)}}^{n-1} a_j x^j, \ldots, \sum_{j \in \mathcal{I}_{(n-1,n)}}^{n-1} a_j x^j \right) \\ &= \operatorname{circ}\left(p_{(0,n)}(x), p_{(1,n)}(x), \ldots, p_{(n-1,n)}(x) \right). \end{aligned}$$

Circulants results

For a polynomial p to be in $\mathcal{P}_n$ the following are necessary

- The n remainder polynomials of p must be in $\mathcal{P}_1$ (for all $x \geq 0$, $p(x) \geq 0$).
- For a polynomial p to be in $\mathcal{P}_n$, the first n terms of p must be nonnegative.
- For a polynomial p to be in $\mathcal{P}_n$, the last n terms of p must be nonnegative.

These results can be derived from

$$p(xC) = \operatorname{circ}\left(p_{(0,n)}(x), p_{(1,n)}(x), \ldots, p_{(n-1,n)}(x)\right).$$

Jordan block

If $n \in \mathbb{N}$, $n > 1$, and $\lambda \in \mathbb{C}$, then $J_n(\lambda)$ denotes the *Jordan block with eigenvalue* λ, i.e.,

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \in \mathrm{M}_n(\mathbb{R}).$$

Jordan Blocks Results

Lemma

If $p \in \mathcal{P}_n$, then $p, p^{(1)}, p^{(2)}, \ldots, p^{(n-1)} \in \mathcal{P}_1$.

This comes from the following fact

$$p(J(x)) = \begin{array}{c} \\ 1 \\ \vdots \\ k \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & \cdots & k & \cdots & n \end{array} \\ \begin{bmatrix} p(x) & \cdots & \frac{p^{(k-1)}(x)}{(k-1)!} & \cdots & \frac{p^{(n-1)}(x)}{(n-1)!} \\ & \ddots & & \ddots & \vdots \\ & & p(x) & & \frac{p^{(k-1)}(x)}{(k-1)!} \\ & & & \ddots & \vdots \\ & & & & p(x) \end{bmatrix} \end{array}$$

Lemma allowing positive matrices

Lemma

If $p \in \mathbb{R}[x]$, then $p \in \mathcal{P}_n$ if and only if $p(A) \geq 0$ whenever $A > 0$.

Proof.

Follows from the continuity of p and the fact that the set of positive matrices of order n is dense in the set of all nonnegative matrices of order n. □

Subsets

Lemma

For all $n \geq 1$, $\mathcal{P}_{n+1} \subset \mathcal{P}_n$.

- Consider $A \in \mathrm{M}_n$ such that $A \geq 0$, then for $p \in \mathbb{R}[x]$ to be in $\mathcal{P}_{n+1}$ we need that

$$p\,(\mathrm{diag}(A, 0)) \geq 0.$$

- The proof for the subset being strict is more involved, but was shown by Lowey in 2023.

Polynomial with negative coefficients

Theorem

Let $p \in \mathbb{R}[x]$ such that

$$p(x) = \sum_{\substack{k=0 \\ k \neq n}}^{2n} a_k x^k - x^k,$$

then there exists $a_k \geq 0$ such that $p \in \mathcal{P}_n$.

The proof for the existence of this polynomial in $\mathcal{P}_n$ is very involved and was one of the main results of Lowey's 2023 paper.

Convexity and convex combinations.

Definition

Let C be a subset of a real vector space X, then C is **convex** if for all $t \in [0, 1]$ and $x, y \in C$ we have $tx + (1 - t)y \in C$.

Definition

Let C be a real vector space, then a **convex combination** is a linear combination where all the coefficients are nonnegative and sum to 1. That is for $x_1, x_2, \ldots, x_n \in C$ a convex combination is

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

where $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.

Example: A triangle is convex

Cones

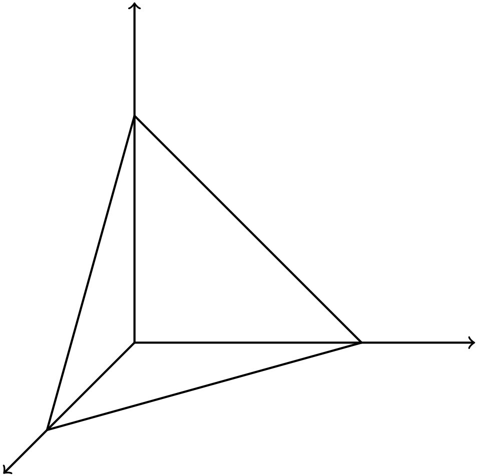
Definition

Let C be a subset of a real vector space X, then C is called a **cone** if it is closed under positive scalar multiplication.

Definition

A cone C is called a **convex cone** if it is closed under convex combinations.

Example: Cone made from three vectors



Polyhedral cones

Definition

Let C be a real vector space, then a **conical combination** is a linear combination where all the coefficients are nonnegative. That is for $x_1, x_2, \ldots, x_n \in C$ a convex combination is

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

where $\alpha_i \geq 0$.

Definition

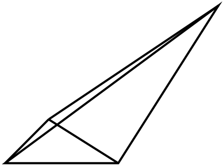
A cone C is polyhedral if it is the conical combination of finitely many vectors (this property is called finitely-generated).

Example: The polyhedral cone of a matrix

Let

$$A = \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

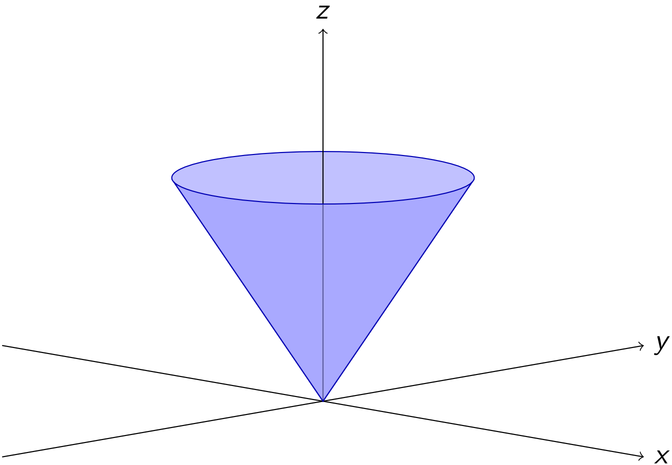
then we can generate a polyhedral cone by taking conical combinations of the columns of A.



Non-polyhedral cones

A cone is called non-polyhedral if it is not a polyhedral cone. In particular for this talk that means that the cone is not able to be generated by a finite number of vectors.

Example: Ice cream cone



Extremal vectors and faces of a cone

Definition

An extremal vector is a vector that can't be written as the conical combination of two or more other vectors in the cone.

Definition

Let C be a cone and $F \subseteq C$ also be a cone, then F is a **face** of C if for all $x \in F$ we have that $y \in C$ and $x - y \in C$ implies $y \in F$.

The cone generated by $\mathcal{P}_n$

The set $\mathbb{R}[x]$ forms a vector space and $\mathcal{P}_n \subseteq \mathbb{R}[x]$ forms a convex non-polyhedral cone.

- $\mathcal{P}_n$ is non-polyhedral since the degree of the polynomials can be arbitrary giving a non-finite number of generators.
- $\mathcal{P}_n$ is convex since if $p(A) \geq 0$ and $q(A) \geq 0$, then $tp(A) + (t-1)q(A) \geq 0$ for all $t \in [0,1]$.

When is $\mathcal{P}_{n,m}$ polyhedral?

- For $m < 2n$ we know that $\mathcal{P}_{n,m}$ is the $m+1$ degree nonnegative orthant which gives that $\mathcal{P}_{n,m}$ is polyhedral. In particular it is generated by $\{1, x, x^2, \ldots, x^m\}$.
- However as in the previous slide as m goes to infinity $\mathcal{P}_{n,m}$ becomes non-polyhedral.
- So when does that switch occur?

Face $\mathcal{P}_{n,2n}$ is a face of $\mathcal{P}_{n,m}$

Lemma

Let $m > 2n$, then $\mathcal{P}_{n,2n}$ is a face of $\mathcal{P}_{n,m}$

Lowey's proved this in his 2023 paper. The idea is that $\mathcal{P}_{n,2n}$ forms a subspace of $\mathcal{P}_{n,m}$.

The cone generated by $\mathcal{P}_{2,4}$ is non-polyhedral

Theorem

$\mathcal{P}_{2,4}$ is non-polyhedral.

- This was the other main result of Lowey's 2023 paper. The proof is very long and relies heavily on the full characterization of $\mathcal{P}_2$.
- This gives the conjecture that $\mathcal{P}_{2,2n}$ is non-polyhedral for all $n \geq 1$.

Possible extremal generators of $\mathcal{P}_{n,2n}$

- Polynomials with nonnegative coefficients are always in $\mathcal{P}_{n,2n}$, so the set $\{1, x, x^2, \ldots, x^{2n}\}$ forms some of the extremal generators of $\mathcal{P}_{n,2n}$.
- From Lowey's 2023 paper we know that there always exist polynomials whose first and last n terms are nonnegative and whose middle x^n term has negative coefficients. So for $\mathcal{P}_{n,2n}$ to be polyhedral we need the set of extremal polynomials that generate those to be finite.

Mapping positive matrices to nonnegative matrices

Definition

Let $p, q \in \mathbb{R}[x]$, then define

$$g_{p,q,t}(x) = tp(x) + (1-t)q(x).$$

Time to wiggle

Lemma

Let $X = \{x, x^2, \ldots, x^n\}$ *and* $P = \{p_i\}_{i=1}^m \subset \mathcal{P}_{n,2n}$ *where*

$$p_i(x) = \sum_{\substack{k=0 \\ k \neq n}}^{2n} a_{i,k} x^k - x^n.$$

Then if $g_{p,q,t}$ *is not extremal for any* $t \in (0,1)$*,* $p \in P$*, and* $q \in X$*.*

Time to wiggle cont.

- Any polynomial in X maps positive matrices to positive matrices.
- Any polynomial in P maps positive matrices to nonnegative matrices.
- So if we take a convex combination of the two, then we map positive matrices to positive matrices.
- This means that we can make the negative term in the polynomial from P more negative until the convex combination maps positive matrices to nonnegative matrices.
- Thus this convex combination is not extremal.

The cone generated by $\mathcal{P}_{n,2n}$ is non-polyhedral

Theorem

For all $n \geq 2$, $\mathcal{P}_{2,2n}$ is non-polyhedral.

- Assume for contradiction that $\mathcal{P}_{n,2n}$ is a polyhedral cone.
- We take X and P from the previous slides as the sets that are all the extremal vectors for $\mathcal{P}_{n,2n}$.
- By the previous lemma the line connecting polynomials from X and P can't be extremal.
- This implies that we need to add additional extremal vectors to P.
- Continuing this process gives that P can't be a finite set.

Next steps

- Can we get any ideas on what these polynomials in $\mathcal{P}_{n,2n}$ with negative coefficients look like?
- Our guess is that there exist some number of quadratic form inequalities of the coefficients of the polynomial that determine if it is in $\mathcal{P}_{n,2n}$.
- If those quadratic forms exist, can we extend the characterization to $\mathcal{P}_{n,2n+k}$?

Interesting papers

- R. Lowey.
  Proof of a conjecture on polynomials preserving nonnegative matrices
  Linear Algebra Appl., 676:167-276, 2023.
- B. J. Clark and P. Paparella.
  Polynomials that preserve nonnegative matrices.
  Linear Algebra Appl., 637:110–118, 2022.
- R. Loewy and D. London.
  A note on an inverse problem for nonnegative matrices.
  Linear and Multilinear Algebra, 6(1):83–90, 1978/79.

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It's about the cones

Questions?