Polynomials that preserve nonnegative matrices

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January, 2024

Table of Contents

Nonnegative Inverse Eigenvalue Problem (NIEP)

Given a finite list $\Lambda = \{s_1, ..., s_n\}$ of complex numbers, the NIEP asks for necessary and sufficient conditions such that Λ is the spectrum of an n-by-n entrywise-nonnegative matrix.

Background on the problem

In pursuit of a solution to the NIEP, Loewy and London [MR0480563] posed the problem of characterizing all polynomials that preserve all nonnegative matrices of a fixed order.

Notation

- $M_n(\mathbb{R})$ denotes the set of all n-by-n real matrices.
- $\mathbb{R}[t]$ is defined as all polynomials of a finite degree with real coefficients
- $\mathcal{P}_n := \{p \in \mathbb{R}[t] : p(A) \geq 0, \forall A \geq 0, A \in M_n(\mathbb{R})\}$
- The first n terms is defined as terms $\{0, 1, ..., n\}$ of the polynomial.
- The last n terms is defined as terms $\{m, m-1, ..., m-n\}$, where m is the degree of the polynomial.
- The set $\langle m \rangle$ are the natural numbers from 0 to m inclusive.

Remainder polynomials

The polynomial

$$p_{(r,n)}(x) := \sum_{k \equiv r \bmod n} a_k x^k,$$

is called the r mod n *part of* p or the r mod n remainder polynomials.

Circulant matrices

- A circulant is a matrix of the form

$$A = \operatorname{circ}(a_0, a_1, \ldots, a_{n-1}) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}.$$

- There is a special type of circulant called the fundamental circulant or push circulant which has the following form

$$C := \operatorname{circ}(0, 1, 0, \ldots, 0) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

Fundamental circulant

- Any circulant matrix can be decomposed into a polynomial made with the fundamental circulant. Let $A = \mathrm{circ}(a_0, a_1, \ldots, a_{n-1})$, then

$$A = p_A(C) = a_{n-1}C^{n-1} + a_{n-2}C^{n-2} + \cdots + a_1 C + a_0 I.$$

- The fundamental circulant forms a cycle of length n, that is $C^m = C^{nq+r} = C^r$ for any $m \in \mathbb{N}$, $r \in \langle n-1 \rangle$.

Polynomials and the fundamental circulant

Using the fact that the fundamental circulant forms cycles, we can "decompose" our polynomial into it's n remainder polynomials.

$$\begin{aligned}
p(xC) &= \sum_{j=0}^{m} a_j x^j C^j \\
&= \sum_{j=0}^{m} a_j x^j C^{j \bmod n} \\
&= \operatorname{circ}\left(\sum_{j \equiv 0 \bmod n}^{n-1} a_j x^j, \sum_{j \equiv 1 \bmod n}^{n-1} a_j x^j, \ldots, \sum_{j \equiv n-1 \bmod n}^{n-1} a_j x^j \right) \\
&= \operatorname{circ}\left(p_{(0,n)}(x), p_{(1,n)}(x), \ldots, p_{(n-1,n)}(x) \right).
\end{aligned}$$

Circulants Results

For a polynomial p to be in $\mathcal{P}_n$ the following are necessary

- The n remainder polynomials of p must be in $\mathcal{P}_1$.
- For a polynomial p to be in $\mathcal{P}_n$, the first n terms of p must be nonnegative.
- For a polynomial p to be in $\mathcal{P}_n$, the last n terms of p must be nonnegative.

These results can be derived from

$$p(xC) = \operatorname{circ}\left(p_{(0,n)}(x), p_{(1,n)}(x), \ldots, p_{(n-1,n)}(x)\right).$$

Jordan Block

If $n \in \mathbb{N}$, $n > 1$, and $\lambda \in \mathbb{C}$, then $J_n(\lambda)$ denotes the *Jordan block with eigenvalue* λ, i.e.,

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \in \mathsf{M}_n(\mathbb{R}).$$

Jordan Blocks Results

- If $p \in \mathcal{P}_n$, then $p, p^{(1)}, p^{(2)}, \dots, p^{(n-1)} \in \mathcal{P}_1$

This comes from the following fact

$$p(J(x)) = \begin{array}{c c} & \begin{array}{ccccc} 1 & \cdots & k & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ k \\ \vdots \\ n \end{array} & \begin{bmatrix} p(x) & \cdots & \frac{p^{(k-1)}(x)}{(k-1)!} & \cdots & \frac{p^{(n-1)}(x)}{(n-1)!} \\ & \ddots & & \ddots & \vdots \\ & & p(x) & & \frac{p^{(k-1)}(x)}{(k-1)!} \\ & & & \ddots & \vdots \\ & & & & p(x) \end{bmatrix} \end{array}$$

Lemma allowing positive matrices

Lemma

If $p \in \mathbb{R}[x]$, then $p \in \mathcal{P}_n$ if and only if $p(A) \geq 0$ whenever $A > 0$.

Proof.

Follows from the continuity of p and the fact that the set of positive matrices of order n is dense in the set of all nonnegative matrices of order n. $\square$

Lemma 2x2 matrix similar form

Lemma

Let $A \in \mathrm{M}_2(\mathbb{R})$ and suppose that $A > 0$. If $\sigma(A) = \{\rho, \mu\}$, with $\rho > |\mu|$, then A is similar to a matrix of the form

$$\frac{1}{1+\alpha}\begin{bmatrix} \alpha\rho + \mu & \rho - \mu \\ \alpha(\rho - \mu) & \alpha\mu + \rho \end{bmatrix},$$

where $\alpha > 0$.

Proof

By the Perron–Frobenius theorem for positive matrices, there is a positive vector x such that $Ax = \rho x$. If $D = \operatorname{diag}(x_1, x_2)$, then the positive matrix

$$B := D^{-1}AD$$

has row sums equal to ρ. Thus, there is an invertible matrix $\hat{S} = \begin{bmatrix} 1 & \hat{a} \\ 1 & \hat{b} \end{bmatrix}$ such that

$$B = \hat{S} \begin{bmatrix} \rho & 0 \\ 0 & \mu \end{bmatrix} \hat{S}^{-1}.$$

Proof cont.

If

$$S := \hat{S} \begin{bmatrix} 1 & 0 \\ 0 & 1/\hat{a} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix},$$

where $a = \hat{b}/\hat{a}$, then

$$B = S \begin{bmatrix} \rho & 0 \\ 0 & \mu \end{bmatrix} S^{-1}.$$

Proof cont.

Furthermore, $a < 0$ (otherwise,

$$B = \frac{1}{1-a}\begin{bmatrix} a\rho - \mu & \mu - \rho \\ a(\rho - \mu) & a\mu - \rho \end{bmatrix}$$

and $b_{12} < 0$). Thus,

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -\alpha \end{bmatrix},\ \alpha > 0,$$

and

$$B = \frac{1}{1+\alpha}\begin{bmatrix} \alpha\rho + \mu & \rho - \mu \\ \alpha(\rho - \mu) & \alpha\mu + \rho \end{bmatrix}. \tag{1}$$

Characterization of 2×2 nonnegative matrices

Theorem

If $p \in \mathbb{R}[x]$, then $p \in \mathcal{P}_2$ if and only if $p_e, p_o, p' \in \mathcal{P}_1$ and $yp(x) + xp(-y) \geq 0$ for all $0 < y < x$.

Proof of 2x2 case, necessary

- If $p \in \mathcal{P}_2$, the circulants and Jordan blocks show $p_e, p_o, p' \in \mathcal{P}_1$.
- Let ρ and μ be real numbers such that $0 < \mu \le \rho$ and let

$$A = \begin{bmatrix} 0 & \rho \\ \mu & \rho - \mu \end{bmatrix} = \frac{1}{\rho + \mu} \begin{bmatrix} \rho & 1 \\ -\mu & 1 \end{bmatrix} \begin{bmatrix} -\mu & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \mu & \rho \end{bmatrix},$$

then

$$p(A) = \frac{1}{\rho + \mu} \begin{bmatrix} \rho p(-\mu) + \mu p(\rho) & \rho(p(\rho) - p(-\mu)) \\ \mu(p(\rho) - p(-\mu)) & \rho p(\rho) + \mu p(-\mu) \end{bmatrix} \ge 0.$$

Proof of 2x2 case, sufficient

- If $p_e, p_o, p' \in \mathcal{P}_1$ and $yp(x) + xp(-y) \geq 0$ for all $0 < y < x$, then let A be a positive matrix with spectrum $\{\rho, \mu\}$ by our Lemma we can assume A is similar to a matrix of the form

$$B = \frac{1}{1+\alpha}\begin{bmatrix} \alpha\rho + \mu & \rho - \mu \\ \alpha(\rho - \mu) & \alpha\mu + \rho \end{bmatrix}.$$

- The worst case for α is μ/ρ. Following that we can diagonalize B with diagonal entries ρ and $-\mu$ the polynomial of B can be written as

$$p(B) = Sp(D)S^{-1} = \frac{1}{1+\alpha}\begin{bmatrix} \alpha p(\rho) - p(-\mu) & p(\rho) - p(\mu) \\ \alpha(p(\rho) - p(\mu)) & \alpha p(\mu) + p(\rho) \end{bmatrix}.$$

Ratio condition?

The ratio condition is defined as $yp(x) + xp(-y) \geq 0$ for all $0 < y < x$.

- If the polynomial is totally even, $p = p_e$, then the polynomial always satisfies the ratio condition. This comes from $p_e(-y) = p_e(y)$.
- If the polynomial is totally odd, $p = p_o$, then the polynomial satisfies the ratio condition if and only if $p'' \in \mathcal{P}_2$. This follows from $p_o(-y) = -p_o(y)$ and the ratio condition becomes $yp(x) - xp(y) \geq 0$ for all $0 < y < x$.

Next steps

- Explore more into what the ratio condition means and look for ways to generalize it past the 2x2 case.
- Investigate the relationship between the NIEP and characterizing $\mathcal{P}_n$.
- Give explicit condition for being in $\mathcal{P}_3$.
- Give explicit conditions for polynomials preserving nonnegative circulants.

Interesting papers

M. Balaich and M. Ondrus.
A generalization of even and odd functions.
Involve, 4(1):91–102, 2011

G. Bharali and O. Holtz.
Functions preserving nonnegativity of matrices.
SIAM J. Matrix Anal. Appl., 30(1):84–101, 2008.

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Linear Algebra Appl., 637:110–118, 2022.

R. Loewy and D. London.
A note on an inverse problem for nonnegative matrices.
Linear and Multilinear Algebra, 6(1):83–90, 1978/79.

Questions?