NIEP, SNIEP, RNIEP, and other problems in spectral nonnegative matrix theory

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March, 2024

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Background

Positive or irreducibly nonnegative

- A matrix is called positive (nonnegative) if all of its entries are positive (nonnegative).
- A nonnegative matrix is called irreducible if its associated directed graph is strongly connected (every vertex can reach every other vertex).

Spectrum

- The spectrum of a matrix is the set (technically multiset) of eigenvalues of that matrix.
- This means that spectral problems and spectral theory are just the study of sets of eigenvalues of matrices.
- Usually when studying spectral problems we are either interested in a class of spectra or a class of matrices who have a given spectra.

Perron-Frobenius theorem

Theorem

Let A be a nonnegative matrix with spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, then

$$\rho(A) = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\} \in \sigma(A).$$

- This is the foundation result to nonnegative matrix analysis.
- The theorem is stronger when dealing with positive or nonnegative irreducible matrices.

Examples

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = W\mathrm{diag}(2,0)W^{-1}$$

$$\begin{bmatrix} \epsilon & 1 \\ 1 & \epsilon \end{bmatrix} = W\mathrm{diag}(\epsilon+1,\epsilon-1)W^{-1}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = W\mathrm{diag}(1,\omega,\omega^2)W^{-1}$$

Spectral properties of nonnegative matrices

- All eigenvalues of real matrices are either real or come in conjugate pairs.
- All eigenvalues of real symmetric matrices are real.
- All irreducible nonnegative matrices and positive matrices are diagonally similar to a stochastic matrix.
- All the eigenvalues of a stochastic matrix have modulus less than or equal to 1.

Where can those eigenvalues be?

Where can those eigenvalues be?

- A natural starting question is given a fixed Perron root, where can any of the remaining eigenvalues of a nonnegative matrix live?
- From the Perron-Frobenius theorem, all the eigenvalues must live in ball of a fixed radius.

Timeline to a solution

- 1936: Romanovsky found that eigenvalues of stochastic matrices can only lie on the unit circle at roots of unity.
- 1937: Kolmogorov posed the problem of finding the location of roots of stochastic matrices.
- 1945-1946: Dmitriev and Dynkin wrote two papers where they give a partial solution to the region of where eigenvalues of stochastic matrices can lie for $n \leq 5$.

Timeline to a solution cont.

- 1951: Karpelevič wrote a followup paper to Dmitriev and Dynkin where he fully solves the problem.
- 1997: Ito wrote a paper which contains the current most simplified form of Karpelevič's theorem.

Arcing your way to a solution

The following Theorem is Ito's simplified version of the Karpelevič region.

Theorem

The region M_n is symmetric relative to the real axis, is included in the unit disc $|z| \leq 1$, and intersects the circle $|z| = 1$ at points $e^{2\pi ia/b}$, where a and b run over relatively prime integers. The boundary of M_n, consists of these points and of curvilinear arcs connecting them in circular order.

Karpelevič region theorem cont.

Theorem

Let the endpoints of an arc be $e^{2\pi i a_1/b_1}$ and $e^{2\pi i a_2/b_2}$ $(b_1 < b_2)$. Each of these arcs is given by the following parametric equation:

$$\lambda^{b_2}(\lambda^{b_1} - s)^{[n/b_1]} = (1 - s)^{[n/b_1]}\lambda^{b_1[n/b_1]},$$

where the real parameter s runs over the interval $0 \leq s \leq 1$ and where $[n/m]$ is the second to last step of remainders in Euclid's algorithm.

Karpelevič arcs for $n = 3, 4$

For $n = 3$ the arcs are given by

$$\begin{aligned}(\lambda - s)^3 &= (1 - s)^3 \\ \lambda^3 &= s + (1 - s)\lambda.\end{aligned}$$

For $n = 4$ the arcs are given by

$$\begin{aligned}(\lambda - s)^4 &= (1 - s)^4 \\ \lambda^4 &= s + (1 - s)\lambda \\ (\lambda^2 - s)^2 &= (1 - s)^2\lambda\end{aligned}$$

Karpelevič region $n = 3$

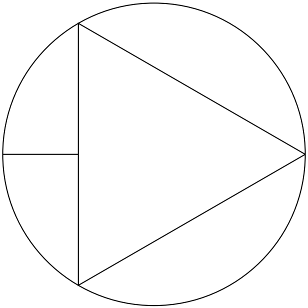

Figure: Karpelevič region of $n = 3$

Karpelevič region $n = 4$

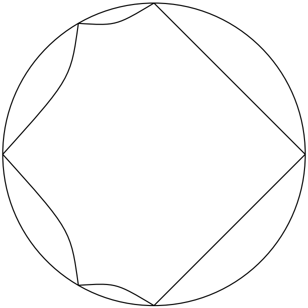

Figure: Karpelevič region of $n = 4$, not to scale

What about doubly stochastic matrices?

- Locating the region where spectra of double stochastic matrices live is unsolved.
- In particular it is unsolved for $n \geq 5$ (as we will see is true for most of these problems).

Where are all those eigenvalues?

NIEP

- Given a finite list $\Lambda = \{s_1, ..., s_n\}$ of complex numbers, the nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions such that Λ is the spectrum of an n-by-n nonnegative matrix.
- An identical formulation of the NIEP asks instead for necessary and sufficient conditions for a list of real numbers to be the coefficients of the characteristic polynomial of a nonnegative matrix.

NIEP current status

- The NIEP is solved for $n \leq 4$.
- The solutions for $n = 1, 2, 3$ are fairly easy to derive from conditions involving the trace (sum of the eigenvalues), moment (sums of powers of the eigenvalues), and Perron root conditions.
- Meehan in 1998 and Torre-Mayo et al. in 2005 both solved the $n = 4$ case. Both of them are very long and complicated.

RNIEP

When a problem is hard, give up and pick an easier one.

A simplification of the NIEP is the real NIEP (RNIEP) where we are given a list of real numbers and asked for necessary and sufficient conditions for it to be the spectra of a nonnegative matrix.

RNIEP current status

- This problem is also only solved for $n \leq 4$.
- The solution for $n = 4$ however is substantially easier only requiring the trace to be nonnegative and for a Perron root to exist.
- The trace/Perron conditions are known not to be sufficient for $n \geq 5$.

SNIEP

A further simplification of the NIEP is the symmetric NIEP (SNIEP) where we are given a list of real numbers and asked for necessary and sufficient conditions for it to be the spectra of a symmetric nonnegative matrix.

SNIEP current status, more of the same

- This problem is also only solved for $n \leq 4$.
- The solutions for $n \leq 4$ are equivalent to the RNIEP.
- This raises the question: are the RNIEP and SNIEP different?

SNIEP and RNIEP are different

Egleston, Lenker, and Narayan showed that

$$\left(1, \frac{71}{97}, -\frac{44}{97}, -\frac{54}{97}, -\frac{70}{97}\right)$$

is realizable in the RNIEP, but not in the SNIEP.

Real algebraic geometry to the rescue

Semialgebraic set

Definition

Let $\mathbb{F}$ be a real closed field. A subset S of $\mathbb{F}^n$ is called a semialgebraic set if it is a finite union of sets defined by polynomial equalities of the form

$$\{x \in \mathbb{F}^n : P(x) = 0\}$$

and inequalities of the form

$$\{x \in \mathbb{F}^n : P(x) \geq 0\}.$$

A semialgebraic set is called basic if no unions are needed.

Elementary symmetric functions

Definition

The kth elementary symmetric function of n complex numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, for $k \leq n$ is

$$S_k(\sigma) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j}.$$

Elementary symmetric functions example

For $n = 2$ the elementary symmetric functions are

$$\begin{aligned} S_1(\lambda_1, \lambda_2) &= \lambda_1 + \lambda_2 \\ S_2(\lambda_1, \lambda_2) &= \lambda_1\lambda_2 \end{aligned}$$

For $n = 3$ the elementary symmetric functions are

$$\begin{aligned} S_1(\lambda_1, \lambda_2, \lambda_3) &= \lambda_1 + \lambda_2 + \lambda_3 \\ S_2(\lambda_1, \lambda_2, \lambda_3) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ S_3(\lambda_1, \lambda_2, \lambda_3) &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

Sums of the principal minors

Definition

A minor of a matrix A is the determinant of sub matrix of A. A minor is called principal if the picked rows and columns of the submatrix are the same.

Definition

For a given matrix A, denote $E_k(A)$ to be the sums of the principal minors of size k of A.

Coefficients of the characteristic polynomial

For a given matrix A let

$$p_A(t) = t^n + (-1)a_1 t^{n-1} + \dots (-1)^{n-1} a_{n-1} t + (-1)^n a_n$$

be its characteristic polynomial, then

$$a_k = E_k(A) = S_k(\sigma(A)).$$

Embedding the NIEP (and sub-problems) as basic semialgebraic sets

Using the equality between $E_k(A)$ and $S_k(\sigma)$ we can embed the NIEP as a semialgebraic set in n^3 dimensions. The inequalities are

$$\begin{aligned} a_{ij} &\geq 0 \text{ for all } i, j \in \{1, \ldots, n\} \\ E_k(A) - S_k(\sigma) &= 0 \text{ for all } k \in \{1, \ldots, n\} \end{aligned}$$

Question

What is wrong with the above semialgebraic set?

Embedding the NIEP cont.

- To overcome the fact that the NIEP can have complex eigenvalues we need to use the fact that the elementary symmetric functions maps complex conjugates to real numbers.
- Using this, we can make modified elementary symmetric functions based on the number of conjugate pairs.
- This gives that the NIEP is the union of $\lfloor (n-1)/2 \rfloor$ basic semialgebric sets in n^3 dimensions.

SNIEP's embedding as basic semialgebraic set

The SNIEP is embedded as

$$\begin{aligned} a_{ij} &\geq 0 \text{ for all } i, j \in \{1, \dots, n\} \\ E_k(A) - S_k(\sigma) &= 0 \text{ for all } k \in \{1, \dots, n\} \\ a_{ij} - a_{ji} &= 0 \end{aligned}$$

So the SNIEP and RNIEP are basic semialgebric sets in n^3 dimensions.

Project your way to a solution

- One of the most important properties of a semialgebraic set is that they are closed under projection. This is known as the Tarski–Seidenberg theorem.
- This means that if you can find a semialgebraic set in an embedded space, then we know there exists a semialgebraic set in the lower dimensional space.
- Therefore the NIEP and related subproblems are semialgebric sets, so a finite union/intersection of polynomial inequalities solves the NIEP.

Cylindrical algebraic decomposition

- Collin's algorithm or Cylindrical algebraic decomposition is the algorithm for computing the projections of a semialgebric set.
- It is a major improvement upon Tarski-Seidenberg's result by giving a straight forward approach to computing the projection.
- This means that the NIEP is solvable and the algorithm for solving it already exists. However...

Doubly exponential means computers don't help

- Cylindrical algebraic decomposition has a doubly exponential computing time.
- Computing the solution to the NIEP for $n = 2$ took less than 1 second.
- I stopped my attempt for computing the solution to the NIEP for $n = 3$ after 13 hours. The software claimed to be approximately 2% done.
- The first result that would be interesting would be $n = 4$ and the first new result would be $n = 5$. So even with massive parallelization computers won't be able to brute force the problem.

Concluding remarks

Summary

- Locating one root (other than the Perron root) is a solved problem for stochastic, but not doubly stochastic, matrices.
- The NIEP and most prominent subproblems are open for $n \geq 5$.
- The NIEP is solvable by the unions and intersections of finitely many polynomial inequalities.
- There exists an algorithm to compute both the solution to the NIEP and the matrices that form the boundary, however the algorithm is so slow that it will never be practical.

Next steps and avenues for potential research

- Is there a approximation for CAD that runs in closer to polynomial time? This may give valuable necessary or sufficient conditions if we can bound the error.
- Gröbner basis and ideals as a tool for projecting algebraic varieties.
- Learn more about what makes the NIEP semialgebric set special. Perhaps it has some property that makes projections easier to compute.

Interesting papers/book

Johnson, Marijuán, Paparella and Pisonero.
The NIEP.
Operator Theory, Operator Algebras, and Matrix Theory

Joanne Swift.
Location of characteristic roots of stochastic matrices.
McGi11 University PhD thesis.

Bochnak , Coste , and Roy.
Real Algebraic Geometry.
Springer-Verlag Berlin.

Jokes?

- Patrick:
  - Put your Artin soul into it.
  - The resultant of your hard work.
  - This is certainly isomorphic to a joke.
- Garret:
  - The NIEP is neat.
  - Alternative nonnegative matrix definition [☺].
- Jared:
  - There are a variety of ways to move forward.
  - We are scheming for a solution.

Questions?