The nonnegative inverse eigenvalue problem is solvable and the algorithm to solve it exists. So why is the problem unsolved?

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Background on the NIEP

NIEP

- Given a finite list $\Lambda = \{s_1, ..., s_n\}$ of complex numbers, the nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions such that Λ is the spectrum of an n-by-n nonnegative matrix.
- An identical formulation of the NIEP asks instead for necessary and sufficient conditions for a list of real numbers to be the coefficients of the characteristic polynomial of a nonnegative matrix.

NIEP current status

- The NIEP is solved for $n \leq 4$.
- The solutions for $n = 1, 2, 3$ are fairly easy to derive from conditions involving the trace, reality, moment (sums of powers of the eigenvalues), and Perron root condition.
- Meehan in 1998 and Torre-Mayo et al. in 2005 both solved the $n = 4$ case. Both of them are very long and complicated.

RNIEP

When a problem is hard, give up and make it easier.

A simplification of the NIEP is the real NIEP (RNIEP) where we are given a list of real numbers and asked for necessary and sufficient conditions for it to be the spectra of a nonnegative matrix.

RNIEP current status

- This problem is also only solved for $n \leq 4$.
- The solution for $n = 4$ however is substantially easier only requiring the trace to be nonnegative and for a Perron root to exist.
- The trace/Perron conditions are known not to be sufficient for $n \geq 5$.

SNIEP

A further simplification of the NIEP is the symmetric NIEP (SNIEP) where we are given a list of real numbers and asked for necessary and sufficient conditions for it to be the spectra of a symmetric nonnegative matrix.

SNIEP current status, more of the same

- This problem is also only solved for $n \leq 4$.
- The solutions for $n \leq 4$ are equivalent to the RNIEP.
- This raises the question: are the RNIEP and SNIEP different?

SNIEP and RNIEP are different

Egleston, Lenker, and Narayan showed that

$$\left(1, \frac{71}{97}, -\frac{44}{97}, -\frac{54}{97}, -\frac{70}{97}\right)$$

is realizable in the RNIEP, but not in the SNIEP.

Real algebraic geometry

Semialgebraic set

Definition

Let $\mathbb{F}$ be a real closed field. A subset S of $\mathbb{F}^n$ is called a semialgebraic set if it is a finite union of a finite intersection of sets defined by polynomial equalities of the form

$$\{x \in \mathbb{F}^n : P(x) = 0\}$$

and inequalities of the form

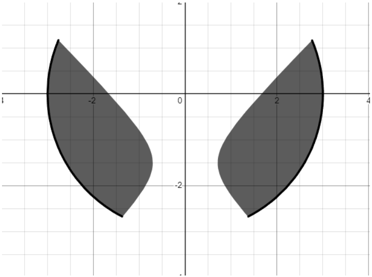
$$\{x \in \mathbb{F}^n : P(x) > 0\}.$$

A semialgebraic set is called basic if no unions are needed.

Example of semialgebraic set

The basic semialgebraic set to the right is defined as the intersection of the following two inequalities.

$$x^2 + y^2 - 9 \leq 0$$
$$x^2 - \left(y + \frac{3}{2}\right)^2 - \frac{1}{2} \geq 0.$$



Elementary symmetric functions

Definition

The kth elementary symmetric function of n complex numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, for $k \leq n$ is

$$S_k(\sigma) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j}.$$

Elementary symmetric functions example

For $n = 2$ the elementary symmetric functions are

$$S_1(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$$
$$S_2(\lambda_1, \lambda_2) = \lambda_1\lambda_2$$

For $n = 3$ the elementary symmetric functions are

$$S_1(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3$$
$$S_2(\lambda_1, \lambda_2, \lambda_3) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$
$$S_3(\lambda_1, \lambda_2, \lambda_3) = \lambda_1\lambda_2\lambda_3$$

Sums of the principal minors

Definition

A minor of a matrix A is the determinant of sub matrix of A. A minor is called principal if the picked rows and columns of the submatrix are the same.

Definition

For a given matrix A, denote $E_k(A)$ to be the sums of the principal minors of size k of A.

Coefficients of the characteristic polynomial

For a given matrix A let

$$p_A(t) = t^n + (-1)a_1 t^{n-1} + \ldots (-1)^{n-1} a_{n-1} t + (-1)^n a_n$$

be its characteristic polynomial, then

$$a_k = E_k(A) = S_k(\sigma(A)).$$

Solvability of the NIEP

Embedding the NIEP as a semialgebraic set

Let $A \in \mathrm{M}_n^+$ with spectra σ. Using the equality between $E_k(A)$ and $S_k(\sigma)$ we can embed nonnegative matrices and their spectra as a semialgebraic set in $\mathbb{R}^{n^3}$. The inequalities are

$$a_{ij} \geq 0 \text{ for all } i, j \in \{1, \dots, n\}$$
$$E_k(A) - S_k(\sigma) = 0 \text{ for all } k \in \{1, \dots, n\}$$

Question

What is wrong with the above semialgebraic set?

Embedding the NIEP cont.

- To overcome the fact that nonnegative matrices can have complex eigenvalues we need to use the fact that the elementary symmetric functions maps complex conjugates to real numbers.
- Using this, we can make modified elementary symmetric functions based on the number of conjugate pairs.
- This gives that the solution to the NIEP is the union of $\lfloor (n-1)/2 \rfloor$ basic semialgebric sets in $\mathbb{R}^{n^3}$

3x3 NIEP embedded semialgebraic set

If $\sigma = (\lambda_1, \lambda_2, \lambda_3)$ is real, then

$$a_{ij} \geq 0 \text{ for all } i, j \in \{1, 2, 3\}$$
$$E_1(A) - \lambda_1 - \lambda_2 - \lambda_3 = 0$$
$$E_2(A) - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 = 0$$
$$E_3(A) - \lambda_1\lambda_2\lambda_3 = 0$$

or if $\sigma = (\lambda_1, \lambda_2 + i\lambda_3, \lambda_2 - i\lambda_3)$, then

$$a_{ij} \geq 0 \text{ for all } i, j \in \{1, 2, 3\}$$
$$E_1(A) - \lambda_1 - 2\lambda_2 = 0$$
$$E_2(A) - 2\lambda_1\lambda_2 - \lambda_2^2 - \lambda_3^2 = 0$$
$$E_3(A) - \lambda_1\lambda_2^2 - \lambda_1\lambda_3^2 = 0$$

SNIEP's and RNIEP's embedding as basic semialgebraic sets

For $A \in \mathrm{M}_n^+$ with spectra σ, the SNIEP is embedded as

$$\begin{aligned} a_{ij} &\geq 0 \text{ for all } i, j \in \{1, \ldots, n\} \\ E_k(A) - S_k(\sigma) &= 0 \text{ for all } k \in \{1, \ldots, n\} \\ a_{ij} - a_{ji} &= 0 \end{aligned}$$

The embedded SNIEP and RNIEP are basic semialgebric sets in $\mathbb{R}^{n^3}$.

Project your way to a solution

- One of the most important properties of a semialgebraic set is that they are closed under projection. This is known as the Tarski–Seidenberg theorem.
- This means that if you can find a semialgebraic set in an embedded space, then we know there exists a semialgebraic set in the projected space.
- Therefore the solution to the NIEP and related subproblems are semialgebric sets. Thus a finite union/intersection of polynomial inequalities solves the NIEP.

The algorithm and its problems

Cylindrical algebraic decomposition

- Collin's algorithm or Cylindrical algebraic decomposition is the algorithm for computing the projections of a semialgebric set.
- It is a major improvement upon Tarski-Seidenberg's result by giving a straight forward approach to computing the projection.
- This means that the NIEP is solvable and the algorithm for solving it already exists. However...

Doubly exponential means computers don't help

- Cylindrical algebraic decomposition has a doubly exponential computing time.
- Computing the solution to the NIEP for $n = 2$ took less than 1 second.
- I stopped my attempt for computing the solution to the NIEP for $n = 3$ after 13 hours. The software claimed to be approximately 2% done.
- The first result that would be interesting would be $n = 4$ and the first new result would be $n = 5$. So even with massive parallelization computers won't be able to brute force the problem.

Concluding remarks

Summary

- The NIEP and most prominent subproblems are open for $n \geq 5$.
- The NIEP is solvable by the unions and intersections of finitely many polynomial inequalities.
- There exists an algorithm to compute both the solution to the NIEP and the matrices that form the boundary, however the algorithm is so slow that it will never be practical.

Next steps and avenues for potential research

- The 5x5 SNIEP has only 10 variables to project in its embedded space. So it seems primed for a brute force approach.
- Finding a bound on the number of polynomials and their degrees for the inequalities in the projected space of the NIEP is crucial for knowing how practical this approach is.
- Learn more about what makes the NIEP semialgebric set special. Perhaps it has some property that makes projections easier to compute.

Interesting papers/book

Johnson, Marijuán, Paparella and Pisonero.
The NIEP.
Operator Theory, Operator Algebras, and Matrix Theory

Joanne Swift.
Location of characteristic roots of stochastic matrices.
McGi11 University PhD thesis.

Bochnak , Coste , and Roy.
Real Algebraic Geometry.
Springer-Verlag Berlin.

Questions?